647510 **MSC4MatC403x** **Seat No :_________**

M.Sc.(Maths) Semester - 4 (*CBCS*) Examination
April/May-2022 (NEW COURSE)
Number Theory -2 (CORE)

Time: 2:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1(A) Answer any Two out of Three. **10 Marks**

(1) If ${}^{p}\!/_{q}$ and ${}^{r}\!/_{s}$ are consecutive fractions in any row, then among all rational fractions with values between these two show that ${}^{(p+r)}\!/_{(q+s)}$ is the unique fraction with smallest denominator.

(2) If n is a positive integer and x is real then show that there is a rational number $\frac{p}{q}$ such that $0 < q \leq n$ and

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{q(n+1)}.$$

(3) Prove that for every real number x there are infinitely many pairs of integers p, q with q positive such that $|qx - p| < \left(\sqrt{5}q\right)^{-1}$.

Q.1(B) Answer any One out of Two. **4 Marks**

(1) Prove that $\sqrt{2} + \sqrt{3}$ is a root of $x^4 - 10\,x^2 + 1 = 0$, and hence establish that it is irrational.

(2) If p and q are positive integers then show that not both the inequalities $\frac{1}{pq} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{p^2} + \frac{1}{q^2}\right)$ and $\frac{1}{p(p+q)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{p^2} + \frac{1}{(p+q)^2}\right)$ can hold.

Q.2(A) Answer any Two out of Three. **10 Marks**

(1) Show that any rational number can be written as a finite simple continued fraction.

(2) If $1 \leq b \leq q_n$, the rational number $\frac{a}{b}$ satisfies

$$\left|x - \frac{p_n}{q_n}\right| \leq \left|x - \frac{a}{b}\right|.$$

(3) Determine the irrational number represented by the infinite continued fraction $x = [3; 6, \overline{1,4}]$.

Q.2(B) Answer any One out of Two. **4 Marks**

(1) Show that the kth convergent of the simple continued fraction $[a_0; a_1, \cdots, a_n]$ has the value $C_k = \frac{p_k}{q_k}, 0 \leq k \leq n.$

(2) Show that $\frac{1+\sqrt{13}}{2} = [2; \overline{3}]$.

Q.3(A) Answer any Two out of Three. **10 Marks**

(1) If p, q is a positive solution of $x^2 - dy^2 = 1$ then $\frac{p}{q}$ is a convergent of the continued fraction expansion of $\sqrt{d}$.

(2) Given the continued fraction expansion $\sqrt{d} = [a_0; a_1, a_2, \cdots]$, define s_k and t_k recursively by the relations

$$s_0 = 0, t_0 = 1$$
$$s_{k+1} = a_k t_k - s_k$$
$$t_{k+1} = \frac{d - s_{k+1}^2}{t_k}, k = 0, 1, 2, \cdots$$

Then show that

(a) s_k and t_k are integers, with $t_k \neq 0$.

(b) $t_k \mid (d - s_k^2)$

(c) $x_k = \frac{s_k + \sqrt{d}}{t_k}$ $for\ k \geq 0$.

(3) Let x_1, y_1 be the fundamental solution of $x^2 - dy^2 = 1$. Then show that every pair of integers x_n, y_n defined by the condition $x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n, n = 1, 2, 3, \cdots$
is also a positive solution.

Q.3(B) Answer any One out of Two. **4 Marks**

(1) If $\frac{p}{q}$ is a convergent of the continued fraction expansion of $\sqrt{d}$ then show that $x = p, y = q$ is a solution of one of the equation $x^2 - dy^2 = k$, $|k| < 1 + 2\sqrt{d}$.

(2) Find the fundamental solution of $x^2 - 29y^2 = 1$.

Q.4(A) Answer any Two out of Three. **10 Marks**

(1) Solve the linear Diophantine equation $72x + 20y = 1000$ by means of continued fraction.

(2) If x, y, z is a primitive Pythagorean triplet then one of the integers x or y is even, while other is odd.

(3) Prove that the radius of the inscribed circle of a Pythagorean tringle is always an integer.

Q.4(B) Answer any One out of Two. **4 Marks**

(1) If x, y, z is a primitive Pythagorean triple, then exactly one of the integers x or y is divisible by 3.

(2) Prove that in a primitive Pythagorean triple x, y, z the product xy is divisible by 12 hence $60/_{xyz}$.

Q.5(A) Answer any Two out of Three. **10 Marks**

(1) Obtain all primitive Pythagorean triplets x, y, z in which $x = 60$.

(2) Find the continued fraction expansion of $\frac{172}{51}$.

(3) Determine infinite continued representation of $\pi = 3.141592653 \cdots$.

Q.5(B) Answer any One out of Two. **4 Marks**

(1) Show that the value of any infinite continued fraction is an irrational number.

(2) If x_0, y_0 is a positive solution of the equation $x^2 - dy^2 = 1$ then prove that $x_0 > y_0$.

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